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SCALE COVARIANT GRAVITATION. V. KINETIC THEORY

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ABSTRACT

In this paper we construct a scale covariant kinetic theory for particles and photons. The mathematical framework of the theory is given by the tangent bundle of a Weyl manifold. The Liouville equation is then derived. Solutions corresponding to equilibrium distributions are presented and shown to yield thermodynamic results identical to the ones obtained previously.

Subject headings: cosmology — gravitation — hydrodynamics

I. INTRODUCTION

A scale covariant theory of gravitation has recently been proposed (Canuto *et al.* 1977, hereafter Paper I), which accounts for a possible relative variation of gravitational and electrodynamical clocks by introducing a local scaling function β. The theory provides a framework for studying gravitational phenomena using measuring instruments whose governing dynamical laws are nongravitational. In particular, we fixed attention on electrodynamics whose measuring units shall be called the atomic units. Since astrophysics involves more than gravitation alone, we presented and discussed in a later paper (Canuto and Hsieh 1979, hereafter Paper II) the basis of a scale covariant description of astrophysical phenomena. This is accomplished by relating the basic geometrical parameters of the scale covariant theory to the observable quantities such as photon frequencies and particle energies on the microscopic level and the thermodynamic temperature on the macroscopic level. Based on these identifications, thermodynamic relations were derived using mainly the macroscopic notions of energy and momentum conservation in the scale covariant theory. Since thermodynamics is a limiting case of a more fundamental theory, we set out in this paper to construct a kinetic theory from which the thermodynamic relations given in Paper II can be recovered as a limit. Examination of the scale covariant theory at the microscopic level complements the macroscopic approach of the previous papers and represents, we believe, a step forward toward an improved understanding of the relation between gravitational and electrodynamical interactions.

The kinetic theory in scale covariant gravitation is analogous to that in general relativity, which had been worked out and discussed extensively in the literature in the last decade or so (Lindquist 1966; Ehlers 1971). In spite of the apparent similarities, straightforward generalization of the equations of relativistic kinetic theory to co-covariant form would not lead to the correct result because of the following important differences.

The first difference pertains to the underlying mathematical structure of kinetic theories, the tangent bundle of the space-time manifold. For relativistic kinetic theory, it has been established (Sasaki 1958, 1962; Lindquist 1966) that the tangent bundle of a Riemannian manifold is again a Riemannian manifold whose metric is defined in terms of the metric of the space-time manifold. In the scale covariant theory, space-time is assumed to be a Weyl manifold. The mathematical structure of its tangent bundle has not yet been studied in the literature. Hence, in order to derive the desired relativistic kinetic equations in co-covariant form, we must first undertake the task, in § II, of transferring the Weyl structure from the space-time manifold to its tangent bundle. Only then will the derivation of the co-covariant kinetic equations, such as the Liouville equation, be possible.

The second difference between the general relativistic and scale covariant kinetic theory has to do with the physical content of the Liouville equation. It is well known that in standard kinetic theory the Liouville equation expresses the conservation of the number of particles, locally in phase space, in the absence of interaction. This microscopic conservation law invariably implies global particle number conservation. However, the scale covariant theory of gravitation has been formulated in such a manner that one can accommodate the possibility of nonconservation of the particle number (see Paper I for a discussion of the continuity equation). In this paper we wish to preserve this possibility in the formal development of the kinetic theory. Therefore, an appropriate modification of the microscopic particle conservation law is given in § III, in which we also give a derivation of the Liouville equation which parallels the one in general relativity as described, for example, by Ehlers (1971). The standard particle conservation law is recovered if a particular choice of the gauge function β is made (for details see Papers 1 and II).

An alternative derivation of the Liouville equation is presented in § IV, where we stress the formal aspects of the scale

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covariant theory. In § V we discuss the equilibrium distribution obtained as a solution of the Liouville equation. It will be shown that this distribution leads to the thermodynamic results derived in Paper II. Some final remarks are presented in § VI.

II. THE PHASE SPACE

In developing the mathematical structure of the tangent bundle and in deriving the Liouville equation, we shall often use the compact notation of modern differential geometry. This usage has become quite common in the literature, especially in the field of general relativity and, in fact, has been traditional in relativistic kinetic theory. More importantly, by avoiding complex notations, it is hoped that the basic assumptions and the flow of arguments can come across more clearly.

Let us first define a Weyl manifold. A concise summary of Weyl geometry has been presented in Appendix A of Paper I. A modern mathematical treatment of it can be found in Folland (1970). A tangent vector with nonzero scaling power is then introduced, from which the tangent bundle is constructed. We endow the tangent bundle with a Weyl structure by transferring the structure from the base manifold to the tangent bundle. This is accomplished by making use of the vertical and horizontal lifts of various geometrical structures introduced by Yano and Ishihara (1973). The arbitrariness in the application of the lifts can be removed by requiring that in the limit when the Weyl manifold reduces to the Riemannian one, the resultant structure on the tangent bundle which we have constructed becomes identical to the Riemannian tangent bundle of relativistic kinetic theory.

The space-time manifold, denoted by M, is assumed to be a Weyl manifold. This means that on M are defined a metric g and a 1-form $\aleph$, which we shall call the *metrical potential.* A Weyl connection ∇ can also be defined on M, which satisfies the compatibility condition

$$\nabla g + 2\aleph \otimes g = 0 , \tag{2.1}$$

where $\otimes$ denotes the tensor product. With this combination of g, $\aleph$, and ∇, we can interpret $\aleph$ as giving rise to the differential change of length of a vector, as it is infinitesimally parallel transported, in accordance with the Weyl connection ∇. Under the conformal rescaling,

$$g \to g' = e^{2\varphi} g , \tag{2.2}$$

where φ is an arbitrary function of space and time, the components of the metrical potential transform as

$$\aleph_\mu \to \aleph_\mu{}' = \aleph_\mu + \frac{\partial \varphi}{\partial x^\mu} , \tag{2.3}$$

where $\{x^\mu\}$ is a local coordinate system on M.

The tangent vectors $v(x)$ of M at the point x may in general transform under conformal rescaling as

$$v \to v' = e^{\pi\varphi(x)} v , \tag{2.4}$$

where π is the scaling power of the vector v. We shall consider in particular velocity fields of power -1. In terms of local coordinates, they can be written as

$$v \equiv v^\mu \frac{\partial}{\partial x^\mu} \equiv \frac{dx^\mu}{d\lambda} \cdot \frac{\partial}{\partial x^\mu} , \tag{2.5}$$

where λ is an affine parameter, identifiable with the proper time when massive particles are considered. As usual, we stipulate that the coordinate functions of M are invariant under conformal rescaling.

With the above parameterization of particle path, the equation of motion can be written as

$$\frac{dv^\mu}{d\lambda} + {}^*\Gamma_{\nu\rho}{}^\mu v^\nu v^\rho + (\aleph_\nu v^\nu) v^\mu = a^\mu , \tag{2.6}$$

where ${}^*\Gamma_{\nu\rho}{}^\mu$ are components of the Weyl connections,

$${}^*\Gamma_{\nu\rho}{}^\mu = \Gamma_{\nu\rho}{}^\mu - (g_\nu{}^\mu \aleph_\rho + g_\rho{}^\mu \aleph_\nu - g_{\nu\rho} \aleph^\mu) , \tag{2.7}$$

where $\Gamma_{\nu\rho}{}^\mu$ is the Christoffel symbol for the metric $g_{\mu\nu}$. The term on the right-hand side of equation (2.6) is an acceleration representing nongravitational forces. It is a vector of power -2. For the purposes of this paper, a^μ can be set equal to zero, in which case equation (2.6) coincides with the in-geodesic equation (Paper I, eq. [2.28]).

The tangent bundle, denoted by $T(M)$, consists of the collection of all tangent vectors v of the type described above, at all points of the manifold. Let (x^μ, v^μ) be a set of local coordinates on $T(M)$. A curve in M is a map from the real line

$$C: R \to M \; ; \qquad C(\lambda) = x^\mu(\lambda) .$$

There exists a naturally induced map,

$$\tilde{C}: R \to T(M)\ ; \qquad \tilde{C}(\lambda) = [x^\mu(\lambda), v^\mu(\lambda)]\ ,$$

where $v^\mu(\lambda)$ is given by equation (2.5). $\tilde{C}$ is called the natural lift of the curve C from M to $T(M)$.

The tangent vector to the lifted curve denoted by $\dot{\tilde{C}}$ can be written in terms of the local coordinates as

$$\dot{\tilde{C}} = \frac{dx^\mu}{d\lambda} \cdot \frac{\partial}{\partial x^\mu} + \frac{dv^\mu}{d\lambda}\frac{\partial}{\partial v^\mu}\ . \tag{2.8}$$

Thus, the ensemble of curves on M gives rise to an ensemble of curves on $T(M)$, whose tangent vectors form a vector field on $T(M)$. That is, given a point (x^μ, v^μ) on $T(M)$, a tangent vector to $T(M)$ at that point is defined as

$$\sigma \equiv v^\mu \frac{\partial}{\partial x^\mu} + \frac{dv^\mu}{d\lambda} \cdot \frac{\partial}{\partial x^\mu}\ , \tag{2.9}$$

where $dv^\mu/d\lambda$ is given by equation (2.6). The quantity σ will be called the *phase flow*. When a^μ in equation (2.6) vanishes, σ is the scale covariant analog of the geodesic flow (Linquist 1966).

The tangent bundle $T(M)$ can be adequately considered the phase space of the particles. However, in the literature it is common to consider the phase space to be the sphere bundle $S(M)$, obtainable from $T(M)$ by the restriction

$$g_{\mu\nu} v^\mu v^\nu = 1 \text{ or } 0\ . \tag{2.10}$$

It is convenient to introduce the immersion map i: $S(M) \to T(M)$. If we let (x^μ, v^j) be the local coordinates of $S(M)$, where the index j runs from 1 to 3, the immersion map can be written as

$$i(x^\mu, v^j) = (x^\mu, v^\mu)\ ,$$

with

$$v^0 = v^0[g_{\mu\nu}(x), v^j]$$

given by the solution of equation (2.10). As we shall see below, the differential maps i_* and i^* can be used to relate tangent vectors and differential forms between $S(M)$ and $T(M)$.

It is clear that the set of vectors $\{\partial/\partial x^\mu, \partial/\partial v^\mu\}$ spans the tangent space at any point of $T(M)$. The vectors $\partial/\partial v^\mu$ are said to be *vertical*, because they lie in the kernel of the differential map π_*, where π is the projection from the fiber bundle $T(M)$ onto the base manifold given by $\pi(x^\mu, v^\mu) = x^\mu$. With a connection given on M, we define *horizontal* vectors on $T(M)$ to be the set of vectors spanned by

$$D_\mu = \frac{\partial}{\partial x^\mu} - ({}^*\Gamma_{\mu\rho}{}^\nu v^\rho + \aleph_\mu v^\nu)\frac{\partial}{\partial v^\nu}\ . \tag{2.11}$$

We note that this differs from the standard definition of horizontal vectors by the term $\aleph_\mu v^\nu \partial/\partial x^\nu$. This is because the scaling power π of the tangent vectors has been chosen to be -1. More generally, the term $-\pi\aleph_\mu v^\nu\partial/\partial v^\nu$ would appear in the definition (2.11), so that when the scaling power is zero, one recovers the standard form used in general relativity.

We now proceed to define metrics on the tangent and sphere bundle, given a metric g on the base manifold M. This can be most easily achieved by first defining the vertical and horizontal lifts of $\{\partial_\mu\}$, which spans vectors on M to vectors on $T(M)$ (Yano and Ishihara 1973):

$$\partial_\mu{}^\perp \equiv \frac{\partial}{\partial v^\mu}\ ; \qquad \partial_\mu{}^\parallel \equiv D_\mu\ , \tag{2.12}$$

where we have used the symbols $\perp$ and $\parallel$ to denote vertical and horizontal lifts. Lifts of general vector fields on M can be obtained by linear extension of equation (2.12). Thus,

$$(X^\mu\partial_\mu)^\perp = X^\mu \frac{\partial}{\partial v^\mu}\ , \qquad (X^\mu\partial_\mu)^\parallel = X^\mu D_\mu\ .$$

We note that the set $\{\partial_\mu{}^\perp, \partial_\mu{}^\parallel\}$ spans the tangent space of $T(M)$; hence, a metric on $T(M)$ is completely defined once its action on the above set of vectors is specified. Thus, we define g^D on $T(M)$ by stipulating that

$$\begin{aligned} g^D(\partial_\mu{}^\perp, \partial_\nu{}^\perp) &= g(\partial_\mu, \partial_\nu)\ , \\ g^D(\partial_\mu{}^\parallel, \partial_\nu{}^\parallel) &= g(\partial_\mu, \partial_\nu)\ , \\ g^D(\partial_\mu{}^\parallel, \partial_\mu{}^\perp) &= g^D(\partial_\mu{}^\perp, \partial_\nu{}^\parallel) = 0\ . \end{aligned} \tag{2.13}$$

A metric on $S(M)$ is defined in terms of g^D by way of the differential map i_*. Let u_1 and u_2 be tangent vectors of $S(M)$. We define a metric g^S on $S(M)$ as

$$g^S(u_1, u_2) = g^D(i_* u_1, i_* u_2) . \tag{2.14}$$

The metric g^D, as defined above, is called the diagonal lift of g. When the Weyl manifold reduces to Riemannian, g^D agrees with the metric defined by Sasaki (1958) and Lindquist (1966) on the tangent bundle. The term diagonal is used for the obvious reason that the connection allows a decomposition of the tangent space into a direct sum of vertical and horizontal parts, and the metric g^D gives vanishing products for vectors belonging to different parts of the decomposition.

Continuing the transference of structures of the base manifold to the tangent bundle, we next define a connection $\nabla^{\parallel}$ on $T(M)$, called the horizontal lift of ∇ on M. Again, we exploit the fact that $\{\partial_\mu{}^{\perp}, \partial_\mu{}^{\parallel}\}$ forms a basis of the tangent space at each point of $T(M)$. Hence, $\nabla^{\parallel}$ is completely specified by (Yano and Ishihara 1973)

$$\nabla_{\partial_\mu{}^{\perp}}{}^{\parallel}\partial_\nu{}^{\perp} = 0 \; ; \qquad \nabla_{\partial_\mu{}^{\perp}}{}^{\parallel}\partial_\nu{}^{\parallel} = 0 \; ; \qquad \nabla_{\partial_\mu{}^{\parallel}}{}^{\parallel}\partial_\nu{}^{\perp} = (\nabla_{\partial_\mu}\partial_\nu)^{\perp} \; ; \qquad \nabla_{\partial_\mu{}^{\parallel}}{}^{\parallel}\partial_\nu{}^{\parallel} = (\nabla_{\partial_\mu}\partial_\nu)^{\parallel} . \tag{2.15}$$

The above definition makes sense because the vector $\nabla_{\partial_\mu}\partial_\nu$ is known by virtue of the Weyl connection and can therefore be lifted. The connection $\nabla^{\parallel}$ has the property that the natural lift of geodesic curves of ∇ on M are geodesic curves of $\nabla^{\parallel}$ on $T(M)$.

Having thus fixed the connection and metric on $T(M)$, the metrical potential on $T(M)$ can be determined by using a compatibility condition analogous to equation (2.1). We find that the naturally induced 1-form on $T(M)$ compatible with g^D and $\nabla^{\parallel}$ is the vertical lift of $\aleph$, denoted by $\aleph^{\perp}$. In terms of local coordinates, if we write

$$\aleph = \aleph_\mu dx^\mu , \tag{2.16}$$

then $\aleph^{\perp}$ is defined to be

$$\aleph^{\perp} = \aleph_\mu dx^\mu . \tag{2.17}$$

That is, the vertical lift of $\aleph$ does not induce nonvanishing coefficients to the forms dv^μ on $T(M)$. It can be checked that the following compatibility condition is satisfied.

$$\nabla^{\parallel} g^D + 2\aleph^{\perp} \otimes g^D = 0 . \tag{2.18}$$

III. THE LIOUVILLE EQUATION

In order to bring out the salient features of the derivation of the Liouville equation within the scale covariant theory of gravitation, it is convenient to first recapitulate the derivation in standard relativistic kinetic theory. In the process, we shall introduce the necessary notations.

Let Ω denote the volume element on $S(M)$. We define a 6-form on $S(M)$ by contracting the volume element with σ:

$$\omega \equiv \Omega(\sigma) , \tag{3.1}$$

We remark that even though σ was previously defined on $T(M)$, a well-defined phase flow vector field exists on $S(M)$ as well. With the metrical structures of $T(M)$ and $S(M)$ one can define on $T(M)$ a 1-form dual to σ which can then be pulled back onto $S(M)$ using i^*. This 1-form on $S(M)$ in turn gives a dual vector field on $S(M)$ which we again denote by σ to emphasize the equivalence of the phase flow operator in the two manifolds in the sense described above.

Note that from equation (3.1), we have

$$\omega = * \tilde{\sigma} \tag{3.2}$$

where $*$ is the Hodge operator (see, e.g., Flanders 1963) and $\tilde{\sigma}$ is the 1-form dual to σ in $S(M)$. Hence, ω and $\tilde{\sigma}$ are relative dual forms on $S(M)$. This observation is useful because

$$\text{divergence of } \sigma \equiv \nabla \cdot \sigma = * \, d * \tilde{\sigma} = * \, d\omega . \tag{3.3}$$

Thus, the exterior derivative of ω can be expressed in terms of the divergence of the phase flow vector, which is generally much easier to compute. Sasaki (1962) has shown that in the Riemannian case $\nabla \cdot \sigma = 0$. Hence,

$$d\omega = 0 , \tag{3.4}$$

which states, in compact notation, the relativistic form of the Liouville's theorem (Lindquist 1966).

Next, we introduce a nonnegative scalar function on the sphere bundle, f: $S(M) \to R$. Let Σ be a six-dimensional hypersurface in $S(M)$. The definition of f, called the *distribution function*, is such that

$$\int_\Sigma f\omega = \text{number of flow lines crossing the surface } \Sigma .$$

Given a domain $\mathscr{D}$ of $S(M)$, the net change of flow lines into and out of the region is given by

$$\delta N = \int_{\partial\mathscr{D}} f\omega = \int_{\mathscr{D}} d(f\omega) = \int_{\mathscr{D}} df \wedge \omega = \int_{\mathscr{D}} df \wedge \Omega(\sigma) = \int_{\mathscr{D}} \sigma(f)\Omega\,, \tag{3.5}$$

where $\partial\mathscr{D}$ denotes the boundary of the domain $\mathscr{D}$. In the above, we have used the Stokes theorem and equation (3.4). The last equality was established by Ehlers (1971). If we now stipulate the microscopic particle number conservation law $\delta N = 0$, we find

$$\sigma(f) = 0\,, \tag{3.6}$$

which is the standard relativistic Liouville equation.

We next turn to an analogous derivation, within the framework of the scale covariant theory. It can be shown, after some computations, that for an in-geodesic flow (i.e., $a^\mu = 0$) on a Weyl manifold we have

$$(\nabla \cdot \sigma)_{S(M)} = (\nabla \cdot \sigma)_{T(M)} = 3\aleph_\mu v^\mu\,. \tag{3.7}$$

Consequently,

$$d\omega = (\nabla \cdot \sigma)\Omega\,. \tag{3.8}$$

Thus, with the distribution function defined as before, in place of equation (3.5) we now have

$$\delta N = \int_{\partial\mathscr{D}} f\omega = \int_{\mathscr{D}} d(f\omega) = \int_{\mathscr{D}} (df \wedge \omega + f d\omega) = \int_{\mathscr{D}} [\sigma(f) + (\nabla \cdot \sigma)f]\Omega\,. \tag{3.9}$$

To complete the derivation, a statement must be made concerning δN. In the standard theory by setting δN equal to zero one imposes the conservation of the number of particles, locally in phase space. That is, the number of particles with a given velocity is a constant along the motion of the particles. This microscopic conservation law must be relaxed in the scale covariant theory for the reasons presented before and discussed in Papers I and II. However, any such modification must be compatible with the macroscopic continuity equation already derived (Paper I, eq. [2.47]), namely,

$$(nu^\mu)_{;\mu} = (1 - \pi_g)(\aleph_\mu u^\mu)n\,, \tag{3.10}$$

where u is the fluid velocity, n is the particle number density, and π_g is the power of the gravitational constant G under scaling (see Paper I, eq. [2.44]). If δN now denotes rate of change of the total number of particles within a comoving volume of the fluid, it is easy to show from equation (3.10) that

$$\delta N = (1 - \pi_g)\aleph_\mu u^\mu N\,. \tag{3.11}$$

The form of equation (3.11) easily suggests the following modification of the microscopic particle conservation law:

$$\delta N = \int_{\mathscr{D}} (1 - \pi_g)\aleph_\mu v^\mu f \Omega\,. \tag{3.12}$$

For sufficiently small region $\mathscr{D}$, equation (3.12) reduces to equation (3.11), if in the latter equation we assume there is no velocity dispersion among particles within a fluid element.

Combining equations (3.9) and (3.12), we finally obtain the Liouville equation in the scale covariant theory:

$$\sigma(f) + (\nabla \cdot \sigma)f = (1 - \pi_g)\aleph_\mu v^\mu f\,. \tag{3.13}$$

In the above equation, σ can be considered an operator defined on $T(M)$ even though in the derivation it has been assumed to be an operator on $S(M)$. This is due to the equivalence of the operators in the two spaces as we discussed earlier, as well as to equation (3.7). In fact, it is sometimes advantageous to consider σ in $T(M)$, for the expression of equation (3.13) in terms of local coordinates would be manifestly covariant. In this case the distribution function should be considered to have an implicit δ-function to account for the normalization condition (2.6).

In the process of developing a scale covariant kinetic theory, we have gained a better understanding of the process of particle creation. Equation (3.12) states that if particle creation occurs, the creation rate is proportional to the phase space density of the particles. It remains to be shown that equation (3.12) and the Liouville equation (3.13) lead back to macroscopic equations, such as the continuity equation and the energy momentum conservation equation, in agreement with the ones derived in previous papers. Toward that end, we consider the first two moments of the Liouville equation.

First, we define

$$n^\mu \equiv \int dv v^\mu f\,; \qquad T^{\mu\nu} = \int dv v^\mu v^\nu f\,, \tag{3.14}$$

where dv denotes the fiber over each point of the space-time manifold M. Note that in order for the above integrals to make sense, the variables which are to be integrated over should be vertical functions with respect to the local horizontal structure. Since the integrations involve the volume elements which are derivable from the metric alone, the local product structure is defined with respect to the Riemannian connection. Thus, writing

$$\hat{D}_\mu \equiv \frac{\partial}{\partial x^\mu} - \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} v^\nu \frac{\partial}{\partial v^\lambda}, \tag{3.15}$$

we must have

$$\hat{D}_\mu(v^\nu) = 0 . \tag{3.16}$$

We also introduce the following decomposition of the volume element of $S(M)$:

$$\Omega = \eta \wedge dv \tag{3.17}$$

where η is the volume element on M.

With equation (3.14), we define number flux vector on M as

$$n \equiv n^\mu \partial_\mu \equiv (nu^\mu)\partial_\mu$$

and perform the following computation:

$$\int_V (\nabla \cdot n)\eta = \int_{\partial V} n^\mu dS_\mu = \int_{\partial V \times Q} v^\mu f dS_\mu \wedge dv = \int_{\partial \mathscr{D}} f\omega = \int_{\mathscr{D}} d(f\omega) = \int_{V \times Q} [\sigma(f) + (\nabla \cdot \sigma)f]\eta \wedge dv . \tag{3.18}$$

In the above, V is a small region of space-time M; dS_μ is the 1-form dual to the hypersurface element on ∂V, the boundary of V; Q is the velocity space, that is, a fiber in the local product structure. It has been assumed in the computation that the distribution function vanishes for large velocities, i.e., $f = 0$ on ∂Q. Making use of the Liouville equation (3.13), equation (3.18) gives

$$\nabla \cdot n = \int dv(1 - \pi_g)\aleph_\mu v^\mu f = (1 - \pi_g)\aleph_\mu n^\mu . \tag{3.19}$$

Thus,

$$n^\mu{}_{;\mu} = (1 - \pi_g)\aleph_\mu n^\mu , \tag{3.20}$$

which is exactly equation (3.10); i.e., the equation governing the evolution of the number density.

A similar computation can be performed for $T = T^{\mu\nu}\partial_\mu \otimes \partial_\mu$, yielding

$$T^{\mu\nu}{}_{;\nu} = (2 - \pi_g)\aleph_\nu T^{\mu\nu} - \aleph^\mu T_\nu{}^\nu . \tag{3.21}$$

The details of the computation, analogous to that in equation (3.18), are not displayed. We note, however, that equation (3.16) must be applied in this case because of the existence of an extra factor of v^ν. Equation (3.21) is recognized to be simply

$$T^{\mu\nu}{}_{*\nu} = 0 , \tag{3.22}$$

stating the vanishing of the co-covariant divergence of the energy momentum tensor (Paper II, eq. [2.6a]).

IV. AN ALTERNATIVE DERIVATION OF THE LIOUVILLE EQUATION

In the above derivation, emphasis was placed on the possible violation of the particle number conservation law within a small region of phase space. In this section, we present a more formal derivation which emphasizes the covariance properties of the Liouville equation and illustrates better the structure of the tangent bundle as a Weyl manifold. The objective is to rewrite covariant equations of general relativity, given on a Riemannian manifold, in co-covariant form appropriate for a Weyl manifold. Examples of the use of this method for the derivation of the equation of motion, energy-momentum conservation equation, etc., in the scale covariant theory has been given in Paper I.

In the case of the relativistic Liouville equation which can be written as

$$\nabla_\sigma f = \sigma(f) = 0 , \tag{4.1}$$

the distribution function f is a scalar function on the Riemannian manifold $T(M)$. The geodesic flow σ is a vector on $T(M)$, and $\nabla_\sigma f$ is the covariant directional derivative, so that equation (4.1) is a scalar equation on $T(M)$. We expect the appropriate scale covariant Liouville equation to be a coscalar equation expressing the vanishing of a co-covariant directional derivative of the coscalar function f. To implement this idea, in addition to the knowledge of the Weyl structure on $T(M)$ established in § II, we need to determine the desired physical content of the Liouville equation so that dimensional analysis yields the scaling power of the quantities involved in the Liouville equation.

The last remark can be elaborated on, and in the process we also review briefly the hypotheses of the scale covariant theory of gravitation (Papers I and II). The space-time manifold considered in the theory is an integrable Weyl manifold.

The metrical potential $\aleph$ is given by the logarithmic gradient of the gauge function β. Aside from a scaling of the metric from the Riemannian manifold, no new dynamical field variable has been introduced. Thus, the covariant equations on the Riemannian manifold are considered valid for *gravitational units*, while the co-covariant equations on the integrable Weyl manifold are valid for *atomic units*. The physical contents of the equations considered remain the same. For example, had we stipulated the standard, general relativistic conservation equation for particle numbers, co-covariant generalization of it for an integrable Weyl manifold would still give strict conservation of the number of particles. Therefore, in gravitational units, we chose to stipulate the general covariant rest-mass energy conservation law. The latter is compatible with the basic hypothesis of the scale covariant gravitation, namely, the scaling of the gravitational field equation among different dynamical units, and allows the possibility of particle nonconservation. The co-covaraint form of it is given by equation (3.10) (details can be found in Paper I).

A similar situation exists in kinetic theory. The relativistic Liouville equation merely states the conservation of the number of particles in phase space, a law which we wish to relax in accordance with our hypothesis. In place of it, we stipulate mass-energy conservation in phase space. Specifically, we define a mass-energy distribution function f_m, related to the number density distribution as follows,

$$f_m = mf . \tag{4.2}$$

In the standard theory, since the particle mass is constant, independently of the choice of units, the Liouville equation may just as well be written

$$\nabla_\sigma f_m = 0 . \tag{4.3}$$

That is, when $\aleph$ vanishes identically, equations (4.3) and (4.1) are equivalent. In the scale covariant theory, we assert that equation (4.3) and not equation (4.1) is valid. Consequently, it is the former equation which we wish to generalize on a Weyl manifold. The scaling power of f_m can now be determined by the condition that

$$\int dv v^\mu f_m = \rho^\mu = \rho u^\mu , \tag{4.4}$$

where ρ is the mass density. Simple calculation then gives

$$\pi_{f_m} = \pi_m - 3 = -2 - \pi_g , \tag{4.5}$$

where π_m is scaling power of mass (see Paper II, eq. [2.10]). If we denote the local coordinates on $T(M)$ by y^α, with α running from 0 through 7, the co-covariant generalization of equation (4.3) can be written as

$$0 = \nabla_\sigma {}^*f_m = \sigma^\alpha (f_m)_{*\alpha} = \sigma^\alpha [(f_m)_{,\alpha} - \pi_{f_m} \aleph_\alpha{}^\perp f_m] = \sigma(f_m) - \pi_{f_m} \aleph^\perp(\sigma) f_m . \tag{4.6}$$

Using equation (4.5) it is easy to show that the above can be rewritten as

$$\sigma(f_m) + (\nabla \cdot \sigma) f_m = (1 - \pi_g) k^\perp(\sigma) f_m , \tag{4.7}$$

where we have made use of equation (3.7). We note also that

$$\aleph^\perp(\sigma) = \aleph_\alpha{}^\perp \sigma^\alpha = \aleph_\mu v^\mu \tag{4.8}$$

because of equation (2.16).

In atomic units, particle masses are constant and drop out of equation (4.7). Thus, we arrive at equation (3.13). For the case of zero rest mass particles, m in equation (4.3) can be replaced by Planck's constant divided by a fundamental atomic length, and the same analysis goes through unchanged. The conservation law and its co-covariant generalization then correspond to the adiabatic conservation law for radiation (Paper II).

V. THE EQUILIBRIUM DISTRIBUTION

Before making use of the Liouville equation (3.13), we shall first cast it in a more familiar form, which is also more appropriate for future applications. Using equations (2.8) and (3.7), the Liouville equation can be written as

$$v^\mu \left[\partial_\mu - ({}^*\Gamma_{\mu\nu}{}^\lambda v^\mu + \aleph_\mu g_\nu{}^\lambda v^\nu) \frac{\partial}{\partial v^\lambda} \right] f + 3\aleph_\nu v^\nu f = (1 - \pi_g) \aleph_\nu v^\nu f . \tag{5.1}$$

Furthermore, we introduce the 4-momentum of the particles:

$$p^\mu = mv^\mu \qquad \text{if } v^\mu v_\mu = 1 ;$$
$$p^\mu = hv^\mu \qquad \text{if } v^\mu v_\mu = 0 , \tag{5.2}$$

where m and h are constant and m is the particle mass (see Paper II for a detailed discussion of the massless case). Replacing v^μ with p^μ, the Liouville equation is finally written as

$$p^\mu \left[\partial_\mu - ({}^*\Gamma_{\mu\nu}{}^\lambda + \aleph_\mu g_\nu{}^\lambda) p^\nu \frac{\partial}{\partial p^\lambda} \right] f + (2 + \pi_g) \aleph_\nu p^\nu f = 0 . \tag{5.3}$$

Clearly, when space-time is Riemannian, i.e., $\aleph_\mu = 0$, the above equation reduces to the standard Liouville equation of relativistic kinetic theory.

The Liouville equation governs the evolution of the phase space distribution of particles in the absence of collisions. More general evolution is governed by the Boltzmann equation which has a collision term added to the Liouville equation. However, for a system in equilibrium, the collision term must vanish, and therefore the equilibrium particle distribution must satisfy the Liouville equation.

The exact form of the collision term depends on the physics of the specific collision process, whose details we shall not discuss in this paper. However, we do assume that in each event of collision the standard energy and momentum conservation laws hold. Furthermore, in as much as we are merely giving a description of classical kinetic theory, we shall assume Boltzmann statistics to be valid. Then it can be shown, by standard arguments, that the equilibrium distribution function takes the form

$$f = \exp\,(\alpha - \lambda_\mu p^\mu)\,, \tag{5.4}$$

where α and λ_μ are functions on the space-time manifold.

To make comparison with standard results, we consider for simplicity the gravitation free case, i.e., when the metric in *gravitational units* is flat. Hence, in atomic units we have (Paper II, eq. [2.19])

$$g_{\mu\nu} = \beta^{-2}\eta_{\mu\nu}\,, \tag{5.5}$$

where $\eta_{\mu\nu}$ is the Minkoskian metric. The standard result in this situation is well known: α and λ_μ are constant. Furthermore, in order for f to vanish for large momenta, λ_μ must be timelike.

To determine α and λ_μ in the scale covariant theory, we require that equation (5.4) satisfy equation (5.3). With the metric given by equation (5.5), it can be shown that

$$^*\Gamma_{\mu\nu}{}^\lambda = 0\,,$$

where it is further assumed that

$$\aleph_\mu = -(\ln\beta)_{,\mu}\,, \tag{5.6}$$

corresponding to the assertion that M is an integrable Weyl manifold (Paper I, eq. [2.23]). Thus, the Liouville equation gives

$$[\alpha_{,\mu} + (2+\pi_g)\aleph_\mu]p^\mu - (\lambda_{\nu,\mu} + \aleph_\mu\lambda_\nu)p^\mu p^\nu = 0\,. \tag{5.7}$$

This equation must hold for arbitrary p^μ, and hence terms of different order in p must vanish separately:

$$\alpha_{,\mu} = (2+\pi_g)(\ln\beta)_{,\mu}\,, \tag{5.8}$$

$$\lambda_{(\nu,\,\mu)} = (\ln\beta),\,({}_\mu\lambda_\nu) \tag{5.9}$$

where equation (5.6) has been used as well as the standard notation for symmetrization with respect to the indices. We remark that if $p_\mu p^\mu = 0$, instead of equation (5.9), we need only require

$$\lambda_{(\nu,\,\mu)} - (\ln\beta),\,({}_\mu\lambda_\nu) = \gamma g_{\mu\nu}\,, \tag{5.10}$$

where γ can be found by taking the trace of the equation (5.10),

$$\gamma = \tfrac{1}{4}g^{\mu\nu}[\lambda_{(\nu,\,\mu)} - (\ln\beta),\,({}_\mu\lambda_\nu)]\,. \tag{5.11}$$

It can be easily seen that the second term of equation (5.7) vanishes with the assumption (5.10) if the particles are massless.

The solution of equation (5.8) is immediate. We find

$$e^\alpha = \beta^{2+\pi_g}\,. \tag{5.12}$$

Equation (5.9) has a simple solution if it is further assumed that β is only time dependent. We obtain in that case

$$\lambda_\mu = \beta(1, \mathbf{0})\,. \tag{5.13}$$

Both equation (5.11) and equation (5.12) are determined up to multiplicative constants, a fact which we shall exploit later.

When massless particles are considered, even though a weaker equation (5.10) can be assumed, the only solution for λ_μ with the assumed $g_{\mu\nu}$ and β is again given by equation (5.12). Thus, the expression (5.4) with α and λ_μ given by equations (4.12) and (5.13) gives the equilibrium distribution for both massive and massless particles. The fact that α and λ_μ are not constant but depend on time through β merely indicates the formal difference between the results obtained here and those in standard relativistic kinetic theory. We shall proceed to establish the relation between the equilibrium distribution (5.4) and the thermodynamics described in Paper II. In the process, it is hoped that physical insight can be gained about the distribution function just derived.

It is useful to recall the following results from Paper II. For a gravitationally flat metric, the solution to the in-geodesic equation (eq. [2.6] with $a^\mu = 0$) can be written as

$$v^\mu = \beta\gamma(1, V^i)\,, \tag{5.14}$$

where

$$\gamma = \left[1 - \sum_i (V^i)^2\right]^{-1/2}$$

and V^i are constants. In particular, when all V^is vanish, we have the velocity field of a set of stationary observers, whose covariant components can be written as

$$u_\mu = \beta^{-1}(1, \mathbf{0})\,. \tag{5.15}$$

Furthermore,

$$\epsilon = u_\mu p^\mu \tag{5.16}$$

is the observed particle energy, valid for both massive and massless particle.

Using equation (5.15), we rewrite equation (5.13) as

$$\lambda_\mu = \lambda\beta^2 u_\mu\,, \tag{5.17}$$

so that the equilibrium distribution takes the form

$$f = A\beta^{2+\pi_\theta} e^{-\lambda\beta^2\epsilon}\,, \tag{5.18}$$

where λ and A are constants.

With the aid of the observer velocity u_μ, a local orthonormal frame can be defined with respect to which the metric is Minkowskian. Let P^μ denote the 4-momentum of a particle with respect to this orthonormal frame. The time and space components of P^μ are, respectively, the observed energy and momentum of the particle. For the metric (5.5) under consideration, it can be shown that

$$P^\mu = \beta^{-1} p^\mu\,. \tag{5.19}$$

Note that

$$\eta_{\mu\nu} P^\mu P^\nu = \beta^{-2}\eta_{\mu\nu} p^\mu p^\nu = g_{\mu\nu} p^\mu p^\nu = m^2\,. \tag{5.20}$$

Hence,

$$\sum_i (P^i)^2 + m^2 = \epsilon^2\,, \tag{5.21}$$

where we have identified ϵ with P^0. In terms of the local orthonormal frame, integration over the momentum space is trivial, and we have the number density given by

$$n = \int d\mathbf{P} f\,. \tag{5.22}$$

Next, we introduce the notion of the temperature of an ensemble in terms of the average kinetic energy of an individual particle in the ensemble. For simplicity, we consider the nonrelativistic limit where $\epsilon \approx P^2/2m$.

A simple computation shows that

$$\langle\epsilon\rangle \equiv \frac{1}{n}\int d\mathbf{P}\epsilon f = \tfrac{3}{2}(\lambda\beta^2)^{-1}\,. \tag{5.23}$$

Comparing equation (5.23) with the well-known relation $\langle\epsilon\rangle = \frac{3}{2}kT$ yields the following identification:

$$(\lambda\beta^2)^{-1} = kT\,. \tag{5.24}$$

so that equation (5.18) can be written as

$$f = A\beta^{2+\pi_\theta} e^{-\epsilon/kT}\,. \tag{5.25}$$

Note that equation (5.24) is valid in general, even though the computation leading to it uses a nonrelativistic approximation. Indeed, using equation (5.25) to compute the average kinetic energy per particle of a relativistic system yields $\langle\epsilon\rangle = 3kT$. The coefficient in front of kT differs from the usual value because we have not taken quantum statistics into account.

With the aid of equation (5.22), we can rewrite equation (5.25) as

$$f = \frac{n}{4\pi\bar{\lambda}^2}\left(\frac{1}{kT}\right)^3 \frac{1}{K_2(\bar{\lambda})} e^{-\epsilon/kT} , \tag{5.26}$$

where $\bar{\lambda} = m/kT$ and, as in the rest of the paper, the velocity of light is set equal to unity. K_n is the nth order modified Bessel function of the second kind. In the nonrelativistic and relativistic limits, we obtain the familiar forms:

$$\text{(nonrel)} \qquad f = \frac{n}{(2\pi mkT)^{3/2}} e^{-P^2/2mkT} , \tag{5.27a}$$

$$\text{(rel)} \qquad f = \frac{1}{8\pi}\frac{n}{(kT)^3} e^{-P/kT} . \tag{5.27b}$$

As usual, the latter is applicable to the massless particles.

In spite of the conventional appearance of equations (5.27a, b), *the distributions depend on the gauge function* β *implicitly through the variables n and T*. In fact, we shall show that they lead to the adiabatic scaling laws derived in Paper II from a thermodynamic approach. Equations (5.22) and (5.25) give in the nonrelativistic and relativistic approximations, respectively,

$$n \sim \beta^{2+\pi_g} T^{3/2} , \tag{5.28a}$$

$$n \sim \beta^{2+\pi_g} T^3 , \tag{5.28b}$$

On the other hand, with equations (3.10) and (5.6) we can express the particle density as

$$n = \frac{N_0}{V} \beta^{\pi_g - 1} , \tag{5.29}$$

where N_0 is a constant. Combining equations (5.28) and (5.29), we find

$$\text{(nonrel)} \qquad \beta^3 V T^{3/2} = \text{constant} , \tag{5.30a}$$

$$\text{(rel)} \qquad \beta^3 V T^3 = \text{constant} , \tag{5.30b}$$

which are the adiabatic scaling laws for classical ideal gas and radiation, respectively (Paper II, eqs. [6.15a, b]).

To further demonstrate the agreement of the kinetic theory developed here with the thermodynamic results, we recall that in Paper II a thermodynamic argument was used to derive the spectral distribution of radiative energy (Paper II, eq. [5.17])

$$\rho_\gamma \sim \beta^{\pi_g - 2} \nu^3 F\left(\frac{\nu}{\beta T}\right) , \tag{5.31}$$

where ν is the frequency of the radiation and F is an arbitrary, universal function of its argument. From equation (5.25), the energy density per energy interval is given by

$$\rho_\gamma(\epsilon) d\epsilon = 4\pi\epsilon(P) f[\epsilon(P)] P^2 dP ,$$

which for massless particles reduces to

$$\rho_\gamma(\epsilon) d\epsilon = 4\pi A \beta^{2+\pi_g} \epsilon^3 e^{-\epsilon/kT} d\epsilon . \tag{5.32}$$

If we recall that for radiation, $\epsilon_\nu \sim \nu/\beta$ (Paper II, eq. [3.15]), it can be easily seen that equation (5.32) takes the form of equation (5.31).

VI. CONCLUDING REMARKS

We have succeeded in constructing a scale covariant kinetic theory which leads to an equilibrium distribution that yields the appropriate thermodynamic limit. To understand why a fundamental, microscopic notion such as the equilibrium distribution would be affected by the introduction of the gauge function β as a local scaling between gravitational and electrodynamical units, we need only recall that in the derivation of the equilibrium distribution which maximizes the entropy, as given in standard statistical mechanics texts, two constraints are imposed: the energy and particle number conservation laws. The important role of relaxing the latter law in the scale covariant kinetic theory has already been pointed out. We emphasize here that even when the particle number is strictly conserved (when the value of π_g is chosen to be 1), the equilibrium distribution (5.18) as well as the thermodynamic adiabatic scaling laws (5.30a, b) still do not reduce to the standard forms. This is because the energy-momentum conservation laws are necessarily modified, according to our hypotheses in Paper I, as long as β is not a constant.

A limitation of the theory should be observed. Our description of the photon is given in terms of classical, massless particles (Paper II). With this model, the resultant equilibrium distribution given by equation (5.27b) does not resemble the Planck distribution for an equilibrium radiation. This difficulty underlines the restrictiveness of the classical kinetic theory where only Boltzmann statistics is considered. Discussion of quantum statistics is beyond the scope of this paper. An understanding of the hypotheses of the scale covariant theory in terms of quantum mechanical principles is of fundamental importance for the further development of the theory. For the present, we can only caution that the results given here should be applied to situations where quantum effects are negligible. Nevertheless, we point out that the classical description gives satisfactory results as far as thermodynamic quantities are concerned.

Finally, we emphasize that the process of constructing the scale covariant kinetic theory is simultaneously deductive and inductive. Notably, the hypothesis that the particle creation rate (if such exists) is proportional to the phase space density is an important new element of the theory. However, the hypothesis has not been imposed ad hoc. Care has been taken to ensure its compatibility with previous results. This is representative of our method of systematically extending the classical, macroscopic theory of scale covaraint gravitation as proposed in Paper I. With the present paper we have completed the extension of the theory to classical microscopic phenomena. While we are still far from the elusive goal of a full understanding of the relation between gravitation and electrodynamics, it is important to remark that a *consistent* framework exists which entertains a relative variation of the field strengths of the above mentioned dynamical forces, and which can be extended thus far without internal inconsistency or incompatibility with observation.

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